DATA FUSION II, Radar Trackers

Date: February 9, 2006 **Time:** 5:00 – 7:30 PM
Instructor: Dr. James K Beard
Credits: 1 **Course Code:** 0901-504-04 **Registration Number:** 13459

Today's topics:

1 Homework and Study Problems from Last Week

1.1 Gelb problem 2-1

Show that

$$\dot{P}^{-1} = -P^{-1} \cdot \dot{P} \cdot P^{-1} \tag{1.1}$$

Solution: We begin with

$$P^{-1} \cdot P = I \tag{1.2}$$

and take the derivative of both sides of this equation with respect to time. The result is, using the chain rule,

$$\dot{P}^{-1} \cdot P + P^{-1} \cdot \dot{P} = 0 \tag{1.3}$$

and the result follows from right-multiplying by P^{-1} and moving the second term to the right-hand side.

1.2 Gelb problem 2-2

For the matrix

$$\begin{bmatrix} 3 & -1 & -4 \\ 2 & 2 & 2 \\ -4 & 0 & -2 \end{bmatrix} \tag{1.4}$$

show that the eigenvalues are 2, -4, and 5.

Solution: The characteristic equation is

$$\left\| \begin{bmatrix} 3-\lambda & -1 & -4 \\ 2 & 2-\lambda & 2 \\ -4 & 0 & -2-\lambda \end{bmatrix} \right\| = 0 \tag{1.5}$$

Expanding the determinant by minors about the left column,

$$\begin{aligned} &(3-\lambda)\cdot(2-\lambda)\cdot(-2-\lambda) \\ &-2\cdot(-1)\cdot(-2-\lambda) \\ &+(-4)\cdot((-1)\cdot(2)-(-4)\cdot(2-\lambda)) \\ &=-\lambda^3+3\cdot\lambda^2+18\cdot\lambda-40 \\ &=-(\lambda-2)\cdot(\lambda+4)\cdot(\lambda-5) \end{aligned} \tag{1.6}$$

1.3 Gelb Problem 2-3

Show that *A* is positive definite if, and only if, all of the eigenvalues of A are positive.

Solution: The definition of the property "positive definite" is on page 21:

$$\underline{x}^T \cdot A \cdot \underline{x} > 0 \tag{1.7}$$

for all real $\underline{x}$. Note that the use of a quadratic form means that any real matrix can be replaced by the real symmetric matrix

$$A_{SYM} = \frac{1}{2}\cdot\left(A + A^T\right) \tag{1.8}$$

so that a matrix in a quadratic form is implicitly symmetric. From Gelp equation (2.1-46) page 19 and *Linear Algebra, Crash Course* page 111, an eigenvalue is a number λ such that, for some vector $\underline{x}$,

$$A\cdot\underline{x} = \lambda\cdot\underline{x}$$

The vector $\underline{x}$ is the eigenvector corresponding to λ. From Theorem 7.5, *Linear Algebra, Crash Course* page 112, the eigenvectors of a real, symmetric matrix are real and orthogonal. Thus the set of eigenvectors may be considered as axes of a rotated coordinate system, and any vector is made up of a linear combination of eigenvectors. Thus, any quadratic from satisfies the inequality

$$\underline{x}^T \cdot A \cdot \underline{x} \ge \lambda_{MIN} \cdot \underline{x}^T \cdot \underline{x} \tag{1.9}$$

Thus, if $\lambda_{MIN} > 0$, the matrix is positive definite.

1.4 Gelb Problem 2-4

If *R(t)* is a time-varying orthogonal matrix, and

$$\frac{dR(t)}{dt}\cdot R^T(t)=S(t) \tag{1.10}$$

show that *S(t)* must be skew-symmetric.

Solution: Construct the product

$$R(t)\cdot R^T = I \tag{1.11}$$

where the product is the identity matrix because *R(t)* is orthogonal, or its transpose is its inverse, as defined in the paragraph following equation (2.1-35) on page 18. Taking the time derivative of this equation and using the chain rule,

$$\frac{dR(t)}{dt}\cdot R^T(t)+R(t)\cdot\frac{dR^T(t)}{dt}=0 \tag{1.12}$$

or,

$$\begin{aligned}\frac{dR(t)}{dt}\cdot R^T(t)&=-R(t)\cdot\frac{dR^T(t)}{dt}\\&=\left(\frac{dR(t)}{dt}\cdot R^T(t)\right)^T\end{aligned} \tag{1.13}$$

which shows that the left-hand side of the equation in the problem statement is skew-symmetric. Thus, the righ-hand side, *S(t)*, must be skew-symmetric.

1.5 Gelb Problem 2-5

Show that the matrix

$$A=\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix} \tag{1.14}$$

satisfies the polynomial equation

$$A^2-5\cdot A-2\cdot I=0\,(1.15) \tag{1.16}$$

and find the eigenvalues. Use the results to show the form of the solution in terms of exponential functions.

Solution: The characteristic equation is found from

$$\left\|\begin{bmatrix}1-\lambda & 2\\ 3 & 4-\lambda\end{bmatrix}\right\|=\lambda^2-5\cdot\lambda-2=0 \tag{1.17}$$

The Cayley-Hamilton theorem (bottom of page 19) states that a matrix satisfies its own characteristic equation, so we have proven the polynomial matrix equation.

From the polynomial equation, we can use

$$A^2=5\cdot A+2\cdot I \tag{1.18}$$

to reduce any polynomial or series in *A* to a linear combination of *A* and *I*. The eigenvalues are

$$\lambda=\frac{5\pm\sqrt{33}}{2} \tag{1.19}$$

The remaining equation, to find the closed forms for $a_1(t)$ and $a_2(t)$, where

$$\exp(A\cdot t) = a_1(t)\cdot I + a_2(t)\cdot A \tag{1.20}$$

is done as follows. First for a two-by-two matrix, the Cayley-Hamilton theorem gives us

$$A^2 - (\lambda_1 + \lambda_2)\cdot A + \lambda_1\cdot\lambda_2\cdot I = 0 \tag{1.21}$$

or,

$$A^2 = (\lambda_1 + \lambda_2)\cdot A - \lambda_1\cdot\lambda_2\cdot I \tag{1.22}$$

This equation means that the matrix exponential series as given by Equation (2.1-48) on page 20 can be collapsed to a linear combination of *I* and *A*. Since we are looking at $\exp(A\cdot t)$, we use

$$(A\cdot t)^2 = (\lambda_1 + \lambda_2)\cdot t\cdot(A\cdot t) + \lambda_1\cdot\lambda_2\cdot t^2 \tag{1.23}$$

Note that the eigenvalues of $(A\cdot t)$ are the eigenvalues of *A* multiplied by *t*.

1.5.1 General Scalar Solution

The Cayley-Hamilton theorem tells us that the matrix itself satisfies its characteristic equation, but we know also that both the eigenvalues also satisfy the characteristic equation. Since the characteristic equation is used to collapse the series for the exponential into a polynomial of order *N-1* (a first-order polynomial for two by two matrices), we also have

$$\exp(\lambda_i\cdot t) = a_1(t) + a_2(t)\cdot\lambda_i,\ \ i = 1,2 \tag{1.24}$$

We can write this equation down for both eigenvalues and solve for $a_1(t)$ and $a_2(t)$:

$$\begin{aligned}\exp(\lambda_1\cdot t) &= a_1(t) + a_2(t)\cdot\lambda_1\\ \exp(\lambda_2\cdot t) &= a_1(t) + a_2(t)\cdot\lambda_2\end{aligned} \tag{1.25}$$

is a set of two linear equations in $a_1(t)$ and $a_2(t)$, and can be solved for $a_1(t)$ and $a_2(t)$. The solution is

$$\begin{aligned}a_1(t) &= \frac{\lambda_1\cdot\exp(\lambda_2\cdot t) - \lambda_2\cdot\exp(\lambda_1\cdot t)}{\lambda_1 - \lambda 2}\\ a_2(t) &= \frac{\exp(\lambda_1\cdot t) - \exp(\lambda_2\cdot t)}{\lambda_1 - \lambda_2}\end{aligned} \tag{1.26}$$

These can be substituted into the two linear equations in $a_1(t)$ and $a_2(t)$ to verify that they are correct. We can differentiate the series for $\exp(A\cdot t)$ term by term and show that

$$\frac{d}{dt}\exp(A\cdot t) = A\cdot\exp(A\cdot t) \tag{1.27}$$

and use this to verify the solution we have for $a_1(t)$ and $a_2(t)$. Note that $a_1(t)$ and $a_2(t)$ as we have found them are the same as given in the problem statement when it gives is “the form $a_1(t)$ and $a_2(t)$ (a hint that there may be a sign difference or a proportionality factor between the equations given there and the exact solution as we have found here), but the sign of the form for $a_1(t)$ given in the problem statement is incorrect.

1.5.2 General Matrix Solution

We can extend the result to higher order matrices by simply writing the set of two linear equations in equations in $a_1(t)$ and $a_2(t)$ as a set of N linear equations in a set of N time functions:

$$\exp\left(\lambda_i \cdot t\right) = \sum_{j=1}^{N} a_j\left(t\right) \cdot \lambda_i^{j-1},\ i = 1, 2 \ldots N \tag{1.28}$$

This set of equations can be written as a vector equal to a matrix times a vector:

$$\begin{bmatrix} 1 & \lambda_1 & \lambda_1^2 & \ldots & \lambda_1^{N-1} \\ 1 & \lambda_2 & \lambda_2^2 & \ldots & \lambda_2^{N-1} \\ 1 & \lambda_3 & \lambda_3^2 & \ldots & \lambda_2^{N-1} \\ \vdots & & & \ddots & \vdots \\ 1 & \lambda_N & \lambda_N^2 & \ldots & \lambda_N^{N-1} \end{bmatrix} \cdot \begin{bmatrix} a_1\left(t\right) \\ a_2\left(t\right) \\ a_3\left(t\right) \\ \vdots \\ a_N\left(t\right) \end{bmatrix} = \begin{bmatrix} \exp\left(\lambda_1 \cdot t\right) \\ \exp\left(\lambda_2 \cdot t\right) \\ \exp\left(\lambda_3 \cdot t\right) \\ \vdots \\ \exp\left(\lambda_N \cdot t\right) \end{bmatrix} \tag{1.29}$$

The set of $a_i(t)$ are found by left-multiplying this equation by the inverse of the matrix. The solution to part (b) of the problem can easily be seen to be given by this equation for N=2.

The matrix is a special form, its rows being given by successive powers of different constants. The determinant of such a matrix is called a *Vandermonde determinant* and is very famous, and a lot of papers have been written about it. In particular, a closed form for this determinant is (see *Introduction to Matrix Analysis*, Second Edition, by Richard Bellman, page 193)

$$\left|\lambda_i^{j-1}\right| = \prod_{1 \le i < j \le N} \left(\lambda_j - \lambda_i\right) \tag{1.30}$$

Note that the matrix is singular if any two eigenvalues are equal.

1.5.3 Differential Equation Solution

The form of $a_1(t)$ and $a_2(t)$ can be found using differential equation theory as follows.

We take the general form for the matrix exponential and take the derivative with respect to time,

$$\begin{aligned} \frac{d \exp\left(A \cdot t\right)}{dt} &= A \cdot \exp\left(A \cdot t\right) \\ &= \frac{da_1\left(t\right)}{dt} \cdot I + \frac{da_2\left(t\right)}{dt} \cdot A \\ &= A \cdot \left(a_1\left(t\right) \cdot I + a_2\left(t\right) \cdot A\right) \\ &= a_2\left(t\right) \cdot A^2 + a_1\left(t\right) \cdot A \\ &= a_2\left(t\right) \cdot \left(\left(\lambda_1 + \lambda_2\right) \cdot A - \lambda_1 \cdot \lambda_2 \cdot I\right) + a_1\left(t\right) \cdot I \\ &= -\lambda_1 \cdot \lambda_2 \cdot a_2\left(t\right) \cdot I + \left(a_1\left(t\right) + \left(\lambda_1 + \lambda_2\right) \cdot a_2\left(t\right)\right) \cdot A \end{aligned} \tag{1.31}$$

We see that we can define $a_1(t)$ and $a_2(t)$ that will satisfy these equations if they satisfy the differential equations

$$\frac{da_1(t)}{dt} = -\lambda_1 \cdot \lambda_2 \cdot a_2(t)$$
$$\frac{da_2(t)}{dt} = a_1(t) + (\lambda_1 + \lambda_2) \cdot a_2(t) \tag{1.32}$$

Thus we see that $a_1(t)$ and $a_2(t)$ are the two solutions of

$$\frac{d^2 a(t)}{dt^2} - (\lambda_1 + \lambda_2) \cdot \frac{da(t)}{dt} + \lambda_1 \cdot \lambda_2 \cdot a(t) = 0 \tag{1.33}$$

that satisfy the initial conditions we obtain from the definition of $\exp(A \cdot t)$. Laplace transform theory tells us that the form of the solutions is as given in the problem statement.

1.6 Z Transform Analysis of the Alpha-Beta Tracker

1.6.1 Alpha-Beta Tracker

The measurements are a sequence of data points y_i and the states are position and velocity, arranged in a state vector $\underline{x}$:

$$\underline{x} = \begin{bmatrix} \text{position} \\ \text{velocity} \end{bmatrix} \tag{1.34}$$

Extrapolation from last update (needed to prepare values for last update):

$$T_i = t_i - t_{i-1}$$
$$\Phi_i = \begin{bmatrix} 1 & T_i \\ 0 & 1 \end{bmatrix} \tag{1.35}$$
$$\underline{\tilde{x}}_i = \Phi_i \cdot \underline{\hat{x}}_{i-1}$$

Update using new data y_i:

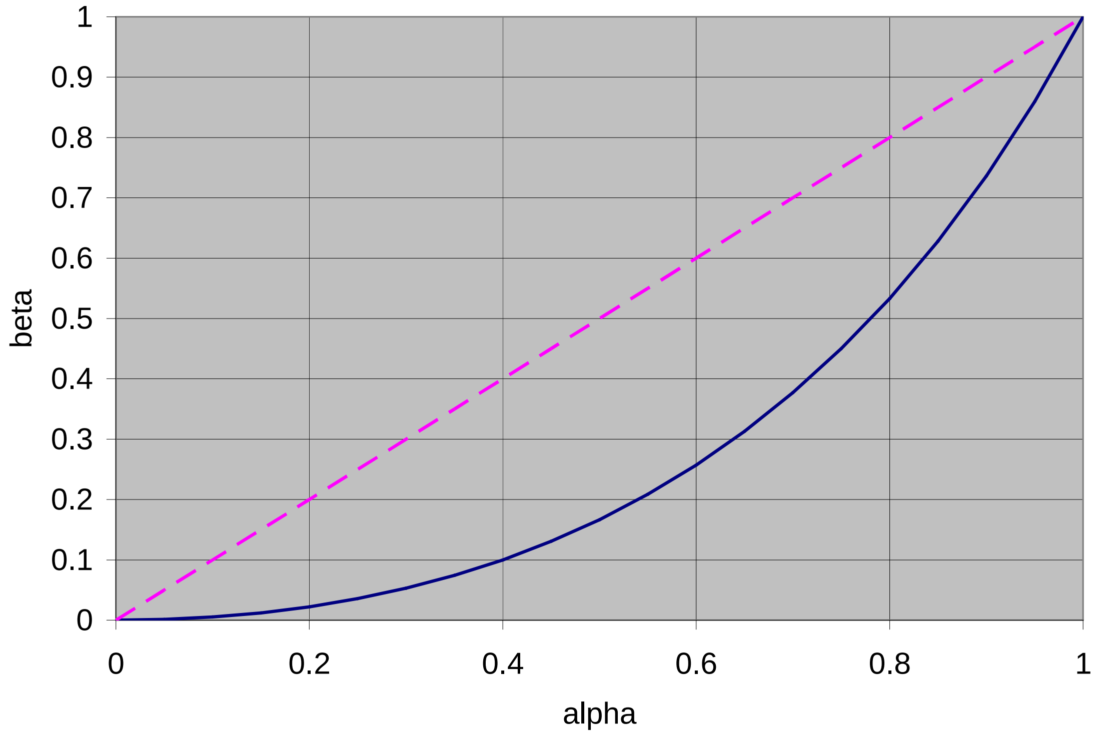
$$K_i = \begin{bmatrix} \alpha & \dfrac{\beta}{T_i} \end{bmatrix}$$
$$h(\underline{x}) = x_1 = \begin{bmatrix} 1 & 0 \end{bmatrix} \cdot \underline{x} \tag{1.36}$$
$$\underline{\hat{x}}_i = \underline{\tilde{x}}_i + K_i \cdot \left(y_i - h(\underline{\tilde{x}}_i) \right)$$

1.6.2 Relationship between α and β

The most commonly used value of β is

$$\beta = \frac{\alpha^2}{2 - \alpha} \tag{1.37}$$

Study problems:
Graph β as a function of α.



When we take $\alpha = 1$ for the second estimate, the result is

$$\underline{\hat{x}}_2 = \begin{bmatrix} y_2 \\ \dfrac{y_2 - y_1}{t_2 - t_1} \end{bmatrix} \tag{1.38}$$

Substituting the extrapolation equation into the update equation and collecting terms gives us this form for the alpha-beta tracker:

$$\underline{\hat{x}}_i = \left(I - K \cdot H\right) \cdot \Phi \cdot \underline{\hat{x}}_{i-1} + K \cdot \underline{y}_i \tag{1.39}$$

If we assume for the purposes of analysis that the time between updates T is constant, we can use Z transforms to analyze this equation, just like any other difference equation. The Z transform of this form for the alpha-beta tracker is

$$\underline{\hat{X}}\left(z\right) = z^{-1} \cdot \left(I - K \cdot H\right) \cdot \Phi \cdot \underline{\hat{X}}\left(z\right) + K \cdot \underline{Y}\left(z\right) \tag{1.40}$$

Collecting terms in $\underline{\hat{X}}\left(z\right)$ we have

$$\left(z \cdot I - \left(I - K \cdot H\right) \cdot \Phi\right) \cdot \underline{\hat{X}}\left(z\right) = z \cdot K \cdot \underline{Y}\left(z\right) \tag{1.41}$$

which gives us the solution

$$\underline{\hat{X}}\left(z\right) = \left(z \cdot I - \left(I - K \cdot H\right) \cdot \Phi\right)^{-1} \cdot z \cdot K \cdot \underline{Y}\left(z\right) \tag{1.42}$$

Thus we see that we can examine the eigenvalues of the matrix

$$
\begin{aligned}
A &= \left(I - K \cdot H\right) \cdot \Phi \\
&= \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} \alpha \\ \dfrac{\alpha^2}{(2-\alpha) \cdot T} \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \end{bmatrix} \right) \cdot \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1-\alpha & (1-\alpha) \cdot T \\ -\dfrac{\alpha^2}{(2-\alpha) \cdot T} & \dfrac{2-\alpha-\alpha^2}{2-\alpha} \end{bmatrix}
\end{aligned}
\tag{1.43}
$$

The characteristic equation, neglecting a denominator of $(2-\alpha)$, is

$$
(2-\alpha) \cdot z^2 - 4 \cdot (1-\alpha) \cdot z + (1-\alpha) \cdot (2-\alpha) = 0 \tag{1.44}
$$

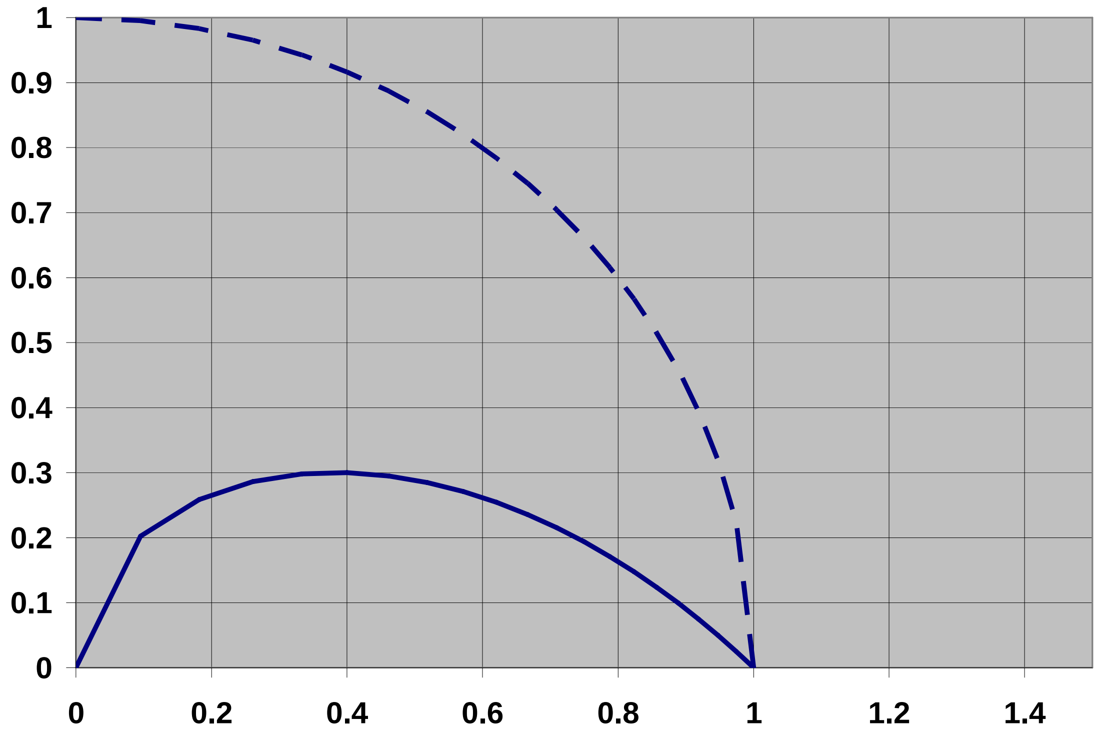
Note that the characteristic values are not a function of the update time *T*. The pole locations are

$$
z_{POLE} = \frac{2 \cdot (1-\alpha) \pm j \cdot \alpha \cdot \sqrt{1-\alpha}}{2-\alpha} \tag{1.45}
$$

The characteristic values as a function of α are as in the table on the next page.

Alpha	Real{z}	Imag{z}
0	1	0
0.05	0.974359	0.024992
0.1	0.947368	0.049931
0.15	0.918919	0.074753
0.2	0.888889	0.099381
0.25	0.857143	0.123718
0.3	0.823529	0.147646
0.35	0.787879	0.171018
0.4	0.75	0.193649
0.45	0.709677	0.215309
0.5	0.666667	0.235702
0.55	0.62069	0.254449
0.6	0.571429	0.271052
0.65	0.518519	0.284848
0.7	0.461538	0.294928
0.75	0.4	0.3
0.8	0.333333	0.298142
0.85	0.26087	0.286264
0.9	0.181818	0.258732
0.95	0.095238	0.202311
1	0	0

The locus of the poles as alpha varies from zero to 1 is shown below.



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2 Example 3.1-2 Page 56

The Schuler loop is the differential equation for an inertial navigation system (INS). A simple form is to look at one axis. An INS usually needs a north-south and an east-west loop, and these loops are coupled, but we can understand some basic principles by examining this simplification.

We define the quantities of interest in terms of differential equations. The quantities are

$$\underline{x} = \begin{bmatrix} \phi \\ \delta v \\ \delta p \end{bmatrix} = \begin{bmatrix} \text{platform tilt angle in radians} \\ \text{system velocity error} \\ \text{system position error} \end{bmatrix} \tag{2.1}$$

We define the errors as

$$\underline{v} = \begin{bmatrix} \varepsilon_g \\ \varepsilon_a \end{bmatrix} = \begin{bmatrix} \text{gryo random drift rate} \\ \text{accelerometer random error} \end{bmatrix} \tag{2.2}$$

From first principles, we have the differential equations

$$\begin{aligned}\frac{d(\delta p)}{dt} &= \delta v \\ \frac{d(\delta v)}{dt} &= \varepsilon_a - g \cdot \sin(\phi) \approx \varepsilon_a - g \cdot \phi \\ \frac{d\phi}{dt} &= \frac{\delta v}{R_e} + \varepsilon_g\end{aligned} \tag{2.3}$$

The block diagram shown as Figure 3.1-4 on page 56 follows from these equations. Given these differential equations, we can collect them and express them in vector-matrix form as

$$\frac{d\underline{x}}{dt} = A \cdot \underline{x} + G \cdot \underline{v}$$

where

$$\begin{aligned}A &= \begin{bmatrix} 0 & \frac{1}{R_e} & 0 \\ -g & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \\ G &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}\end{aligned} \tag{2.4}$$

We can use Laplace transforms to analyze this equation to understand the form of the solutions. The Laplace transform of the differential equation is

$$s \cdot \underline{X}(s) - \underline{x}(0) = A \cdot \underline{X}(s) + G \cdot \underline{V}(s) \tag{2.5}$$

The solution in the *s* domain is

$$\underline{X}(s) = (s \cdot I - A)^{-1} \cdot (\underline{x}(0) + G \cdot \underline{V}(s)) \tag{2.6}$$

Thus we can understand the nature of the solution by examining the eigenvalues of the matrix *A*. The characteristic equation of *A* is

$$s^2 + \frac{g}{R_e} = 0 \tag{2.7}$$

This shows that the INS steady state is an undamped oscillation. You can easily show that this period is about 84 minutes, which is why an INS is sometimes called an 84-minute pendulum. Use the elementary equation for the period of a pendulum as a function of its length to find the length of a “real” 84-minute pendulum.

3 Markov processes are Gaussian noise passed through a recursive filter

See Gelb pp. 42-46 and the lecture.

4 Optimizing the alpha-beta tracker to form a Kalman filter

See Gelb pp. 107-108 and the lecture.

5 Monte Carlo simulations – designing, running, and interpreting them

5.1 Definition of Monte Carlo Simulations

Monte Carlo simulations are

- Computer models of systems or events that are driven in part by unknown, random, or noisy inputs or events
- An ordered set of computer runs of this model with distinct sets of random variables used to represent the random inputs to the system model
- A set of statistical analyses of the events produced by the computer model runs

A simple example is a simulation of a noisy measurement of a quantity, such as we studied in Example 1.0-1 on pages 5 and 6. A more practical example is something that can't be easily analyzed, such as the outputs of a tracker that is adaptive to the inputs. We will use an adaptive alpha-beta tracker and an adaptive Kalman filter in our demonstration to follow.

The set of statistical analyses includes, but is not limited to, the following:

- Mean values of events such as voltage measurements or the tracked ranges of radar measurements.
- The deviations of the measured and estimated ranges (or voltages, or…).
- The maximum peak error in estimated ranges.
- The RMS errors in estimated ranges over the time that the target is tracked.

5.2 Our Demonstration

See the projected computer screen. The file will be available to you by email later today.

We will discuss the results of the simulations.

6 Assignment and homework for next time

Modify the parameters in the simulation and attempt to improve the performance of the simulations. Save the best one and be prepared to discuss it next time.

Gelb, Problem 3-1

Read Gelb, Chapter 3.

Read Gelb, Chapter 4 from page 102 through 110.