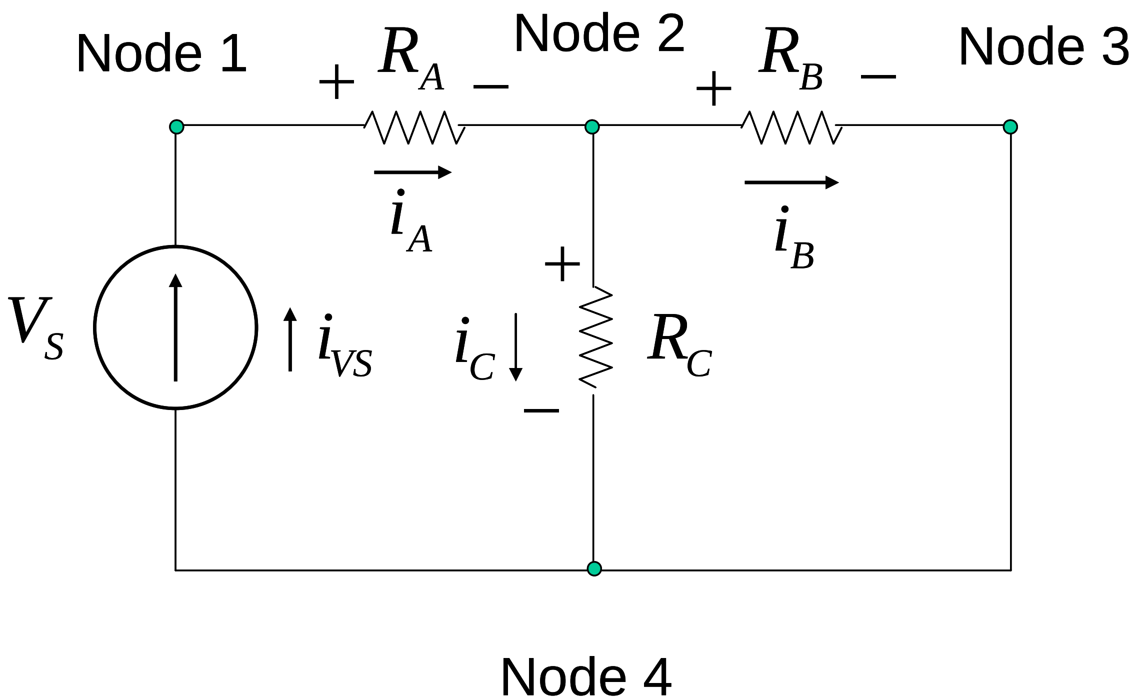
Problem 1 (25%)

Analyze the circuit in the figure using Kirchoff's voltage and current laws, and Ohm's law. Given voltage V_S and resistances R_A, R_B and R_C, find the voltage across each resistor, and the current through the source.



Hint: Write all the equations down and rearrange them as a square matrix of constants and resistances, left-multiplying a vector of unknowns, equal to a vector of knowns. Then, solve the resulting equation by Gaussian elimination.

Solution

We will use the node voltage notation, and embed Ohm's law to minimize the number of unknowns in the problem statement. Kirchoff's current law for Node 1 is incorporated by noting that $i_{VS} = i_A$ and $V_1 = V_S$.

The key insights for working this problem are noting that $V_3 = 0$ and that the only unknown voltage is V_2. Thus we need only i_{VS} and V_2 to solve this circuit .

Type	Equation
Loop 1 (left loop)	$-V_S + i_{VS} \cdot R_A + V_2 = 0$
Node 2	$i_A - i_B - i_C = \frac{V_S - V_2}{R_A} - \frac{V_2}{R_B} - \frac{V_2}{R_C} = 0$

We write these equations in vector-matrix format as

$$\begin{bmatrix} R_A & 1 \\ 0 & \left(\frac{1}{R_A}+\frac{1}{R_B}+\frac{1}{R_C}\right) \end{bmatrix} \cdot \begin{bmatrix} i_{VS} \\ V_2 \end{bmatrix} = \begin{bmatrix} V_S \\ \frac{V_S}{R_A} \end{bmatrix}$$

The solution to this equation is

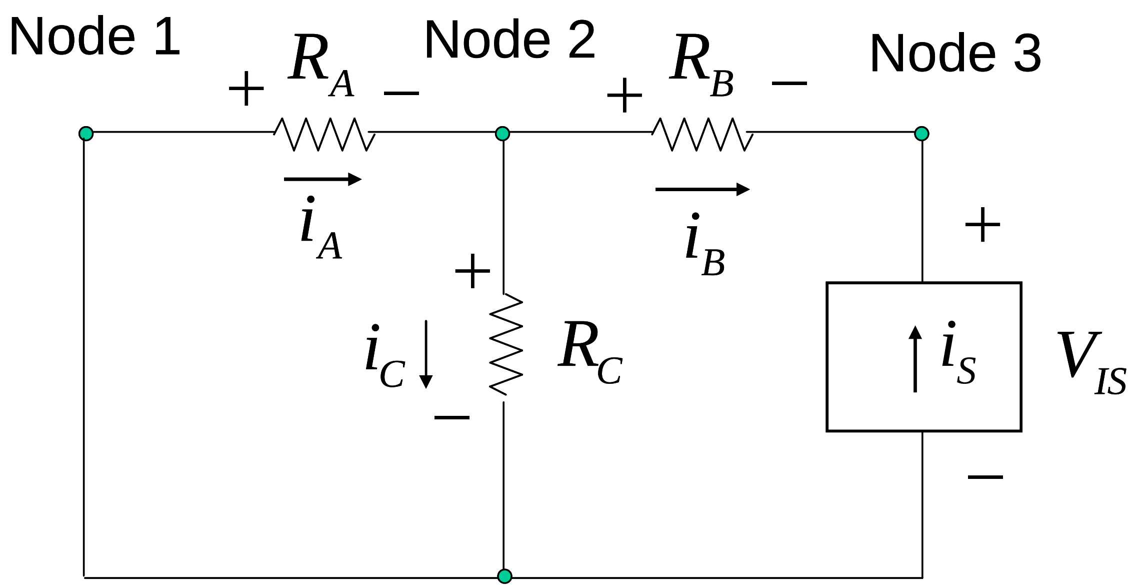
$$\begin{bmatrix} i_{VS} \\ V_2 \end{bmatrix} = \begin{bmatrix} R_A & 1 \\ 0 & \left(\frac{1}{R_A}+\frac{1}{R_B}+\frac{1}{R_C}\right) \end{bmatrix}^{-1} \cdot \begin{bmatrix} V_S \\ \frac{V_S}{R_A} \end{bmatrix} = \frac{1}{1+\frac{R_A}{R_B}+\frac{R_A}{R_C}} \cdot \begin{bmatrix} \left(\frac{1}{R_A}+\frac{1}{R_B}+\frac{1}{R_C}\right) & -1 \\ 0 & R_A \end{bmatrix} \cdot \begin{bmatrix} V_S \\ \frac{V_S}{R_A} \end{bmatrix}$$

or,

$$\begin{bmatrix} i_{VS} \\ V_2 \end{bmatrix} = \frac{V_S}{1+\frac{R_A}{R_B}+\frac{R_A}{R_C}} \cdot \begin{bmatrix} \left(\frac{1}{R_B}+\frac{1}{R_C}\right) \\ 1 \end{bmatrix}.$$

Problem 2 (25%)

Analyze the circuit in the figure using Kirchoff's voltage and current laws, and Ohm's law. Given current i_S and resistances R_A, R_B and R_C, find the voltage across each resistor, and the voltage across the source.



Hint: Write all the equations down and rearrange them as a square matrix of constants and resistances, left-multiplying a vector of unknowns, equal to a vector of knowns. Then, solve the resulting equation by Gaussian elimination.

Solution

This problem is very similar to Problem 1. We see that Node 1 is identical to the ground node, Node 4, which is not named in the figure, and like Node 4, is redundant with the other nodes. Thus we know that $V_1 = 0$. Node 3 can be eliminated as a separate equation by noting that $i_B = -i_S$ and $V_3 = V_{IS}$. Thus we need only V_2 to solve this circuit. We add V_{IS} to the unknowns to simplify the combination of equations by use of matrix representation instead of substitution into scalar equations.

Type	Equation
Ohm's Law for R_B	$\frac{V_2 - V_{IS}}{R_B} = -i_S$
Node 2	$i_A - i_B - i_C = \frac{-V_2}{R_A} + i_S - \frac{V_2}{R_C} = 0$

We write these equations in vector-matrix format as

$$\begin{bmatrix} \frac{1}{R_B} & -\frac{1}{R_B} \\ 0 & \left(\frac{1}{R_A}+\frac{1}{R_C}\right) \end{bmatrix} \cdot \begin{bmatrix} V_{IS} \\ V_2 \end{bmatrix} = \begin{bmatrix} i_S \\ i_S \end{bmatrix}.$$

The solution to this equation is

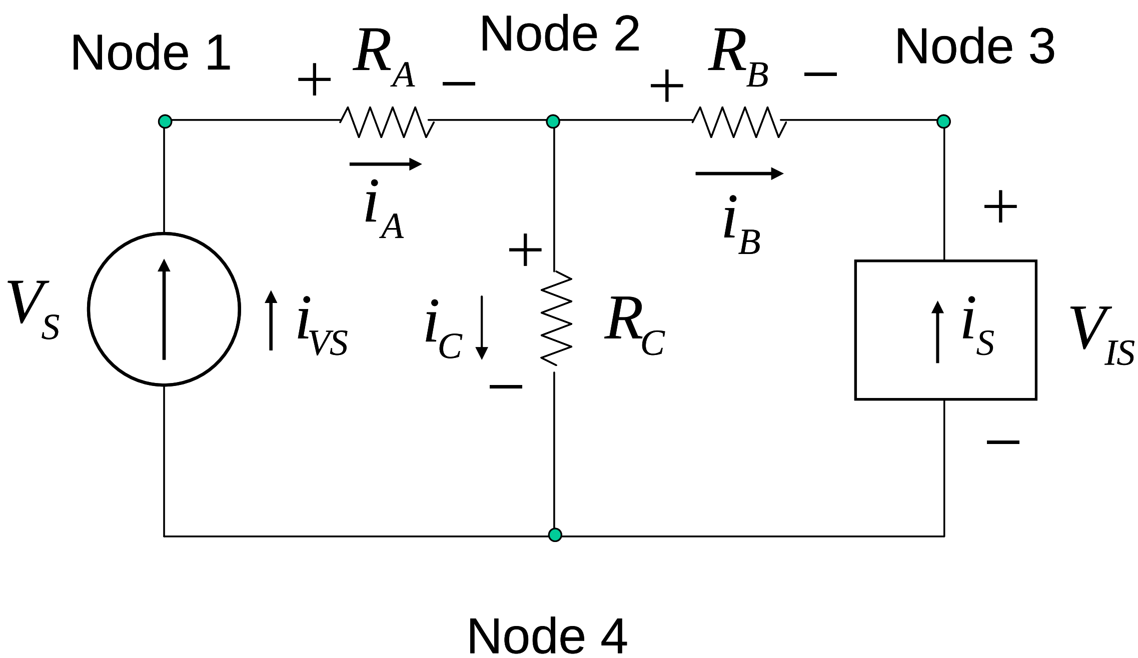
$$\begin{bmatrix} V_{IS} \\ V_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{R_B} & -\frac{1}{R_B} \\ 0 & \left(\frac{1}{R_A}+\frac{1}{R_C}\right) \end{bmatrix}^{-1} \cdot \begin{bmatrix} i_S \\ i_S \end{bmatrix} = \frac{1}{\frac{1}{R_B}\cdot\left(\frac{1}{R_A}+\frac{1}{R_C}\right)} \cdot \begin{bmatrix} \left(\frac{1}{R_A}+\frac{1}{R_C}\right) & \frac{1}{R_B} \\ 0 & \frac{1}{R_B} \end{bmatrix} \cdot \begin{bmatrix} i_S \\ i_S \end{bmatrix}$$

or,

$$\begin{bmatrix} V_{IS} \\ V_2 \end{bmatrix} = \frac{i_S}{\frac{1}{R_B}\cdot\left(\frac{1}{R_A}+\frac{1}{R_C}\right)} \cdot \begin{bmatrix} \frac{1}{R_A}+\frac{1}{R_B}+\frac{1}{R_C} \\ \frac{1}{R_B} \end{bmatrix}.$$

Problem 3 (25%)

Using the principle of superposition, analyze the circuit in the figure using Kirchoff's voltage and current laws, and Ohm's law. Given voltage v_S and current i_S and resistances R_A, R_B and R_C, find the voltage across each resistor, the voltage across the current source, and the current through the voltage source.



Hint: Solve the portion driven by the voltage source by solving a circuit with the current source replaced by an open circuit (i.e., no current flows, or a current source of zero amperes). Then, solve the portion driven by the current source by replacing the voltage source with a short circuit. Pose the solution as the voltage and current across each resistor equal to the sum of that found for each circuit.

Solution

This problem is actually a modification of Problem 1, with the addition of unmodified Problem 2. The current source allows simplification of Node 3 by noting that $i_B = -i_S$. As in Problem 1, we note that $i_{vs} = i_A$ and $V_1 = V_S$. The voltages at Node 2 and Node 3 are unknown but we can write the equations for the voltage at Node 2 and the current through the voltage source i_{VS} as in Problem 1 because the voltage at Node 3 relative to that at Node 2 is given by Ohm's Law for R_B and the source current i_S, and is independent of the rest of the problem. The equations for i_{VS} and V_2 are

Type	**Equation**
Loop 1 (left loop)	$-V_S + i_{VS} \cdot R_A + V_2 = 0$
Node 2	$\frac{V_S - V_2}{R_A} - \frac{V_2}{R_C} + i_s = 0$

We write these equations in vector-matrix format as

$$\begin{bmatrix} R_A & 1 \\ 0 & \left(\frac{1}{R_A} + \frac{1}{R_C}\right) \end{bmatrix} \cdot \begin{bmatrix} i_{VS} \\ V_2 \end{bmatrix} = \begin{bmatrix} V_S \\ \frac{V_s}{R_A} + i_S \end{bmatrix}.$$

The solution to this equation is

$$\begin{bmatrix} i_{VS} \\ V_2 \end{bmatrix} = \begin{bmatrix} R_A & 1 \\ 0 & \left(\frac{1}{R_A} + \frac{1}{R_C}\right) \end{bmatrix}^{-1} \cdot \begin{bmatrix} V_S \\ \frac{V_s}{R_A} + i_S \end{bmatrix} = \frac{1}{\left(1 + \frac{R_A}{R_C}\right)} \cdot \begin{bmatrix} \left(\frac{1}{R_A} + \frac{1}{R_C}\right) & -1 \\ 0 & R_A \end{bmatrix} \cdot \begin{bmatrix} V_S \\ \frac{V_s}{R_A} + i_S \end{bmatrix}$$

or,

$$\begin{bmatrix} i_{VS} \\ V_2 \end{bmatrix} = \frac{1}{\left(1 + \frac{R_A}{R_C}\right)} \cdot \begin{bmatrix} \frac{V_s}{R_C} - i_s \\ V_S + i_S \cdot R_A \end{bmatrix}.$$

Problem 4 (25%)

Solve the circuit of Problem 3 without using superposition by writing all the Kirchoff voltage and current law equations and Ohm's Law for each resistor, and posing the problem as an eight-by-eight matrix of constants and resistances times an eight-vector of unknowns, equal to an eight-vector of knowns. Show that this is the same as the solution to Problem 3.

Solution

Problem 1 can be posed as Problem 3 with the source current set at zero if R_B is taken as infinite, or an open circuit. Problem 2 is Problem 3 with the voltage source at zero volts. Thus, by modifying Problem 1 and Problem 2, we can use the principle of superposition to solve Problem 3 by adding these solutions.

The solution to Problem 1 with R_B taken as infinite is

$$\begin{bmatrix} i_{VS} \\ V_2 \end{bmatrix} = \frac{V_S}{1+\frac{R_A}{R_C}} \cdot \begin{bmatrix} \frac{1}{R_C} \\ 1 \end{bmatrix}.$$

The solution to Problem 2 is

$$\begin{bmatrix} V_{IS} \\ V_2 \end{bmatrix} = \frac{i_S}{\frac{1}{R_B} \cdot \left(\frac{1}{R_A} + \frac{1}{R_C} \right)} \cdot \begin{bmatrix} \frac{1}{R_A} + \frac{1}{R_B} + \frac{1}{R_C} \\ \frac{1}{R_B} \end{bmatrix}.$$

The solution to Problem 3 is

$$\begin{bmatrix} i_{VS} \\ V_2 \end{bmatrix} = \frac{1}{\left(1+\frac{R_A}{R_C}\right)} \cdot \begin{bmatrix} \frac{V_s}{R_C} - i_s \\ V_S + i_S \cdot R_A \end{bmatrix}.$$

We see that the sum of the values for V_2 for Problem 1 and Problem 2 are equal to the value of V_2 as found for Problem 3. Since the value of V_2 agrees, the currents in to Node 2 must agree, so the other circuit voltages and current that are determined by V_2 and Ohm's Law must also agree. Thus we have shown that Problem 3 can be solved by the superposition. Now we must show that a direct solution agrees with this.

We have eight unknowns, and the simplest way to pose the problem is to write the node and loop equations and add Ohm's Law. We can eliminate complexity as in the first three problems by noting that $V_1 = V_S$, $i_B = -i_S$, $i_{VS} = i_A$, and $V_{IS} = V_3$; this type of thing and the voltage node convention allowed Problems 1, 2 and 3 to be solved using two equations, not eight.. This gives us four independent equations:

Type	**Equation**
Loop 1 (left loop)	$-V_S + i_A \cdot R_A + V_2 = 0$
Node 2	$i_A + i_S - i_C = 0$
Ohm's Law for R_A (redundant with Loop 1)	$\frac{V_S - V_2}{R_A} = i_A$
Ohm's Law for R_B	$\frac{V_2 - V_3}{R_B} = -i_S$
Ohm's Law for R_C	$\frac{V_2}{R_C} = i_C$

Note that these equations are not unique, and that simplifications such as folding in the Node 1 voltage identity can be used to represent V_1 as V_S in the other equations and reduce the number of unknowns by one

These equations are, in vector-matrix format,

$$\begin{bmatrix} R_A & 0 & 1 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & -\frac{1}{R_B} & \frac{1}{R_B} \\ 0 & -R_C & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} i_A \\ i_C \\ V_2 \\ V_3 \end{bmatrix} = \begin{bmatrix} V_S \\ i_S \\ i_S \\ 0 \end{bmatrix}.$$

We observe that interchanging the bottom two rows gives us very nearly a lower triangular matrix on the left hand side, so that this operation is most of what is required to complete Gaussian elimination. The equation in this form is

$$\begin{bmatrix} R_A & 0 & 1 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -R_C & 1 & 0 \\ 0 & 0 & -\frac{1}{R_B} & \frac{1}{R_B} \end{bmatrix} \cdot \begin{bmatrix} i_A \\ i_C \\ V_2 \\ V_3 \end{bmatrix} = \begin{bmatrix} V_S \\ i_S \\ 0 \\ i_S \end{bmatrix}.$$

The one in row 1, column 3 is annihilated by subtracting row 3 from row 1. The result of this operation is

$$\begin{bmatrix} R_A & R_C & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -R_C & 1 & 0 \\ 0 & 0 & -\frac{1}{R_B} & \frac{1}{R_B} \end{bmatrix} \cdot \begin{bmatrix} i_A \\ i_C \\ V_2 \\ V_3 \end{bmatrix} = \begin{bmatrix} V_S \\ i_S \\ 0 \\ i_S \end{bmatrix}.$$

The only remaining nonzero upper triangular term in row 1, column 2 is annihilated by multiplying row 2 by R_C and subtracting it from Row 1. The result of this last operation is

$$\begin{bmatrix} R_A + R_C & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -R_C & 1 & 0 \\ 0 & 0 & -\frac{1}{R_B} & \frac{1}{R_B} \end{bmatrix} \cdot \begin{bmatrix} i_A \\ i_C \\ V_2 \\ V_3 \end{bmatrix} = \begin{bmatrix} V_S - R_c \cdot i_S \\ i_S \\ 0 \\ i_S \end{bmatrix}.$$

By direct substitution from the first row, we have

$$i_{VS} = i_A = \frac{V_S - R_C \cdot i_S}{R_A + R_C}.$$

Given i_A, we can use direct substitution in the second row to get i_C, which gives us V_2 by Ohm's Law:

$$V_2 = R_C \cdot i_C = R_C \cdot (i_S + i_A) = R_C \cdot \left(i_S + \frac{V_S - R_C \cdot i_S}{R_A + R_C} \right)$$

or,

$$V_2 = R_C \cdot \left(i_S + \frac{V_S - (R_C + R_A - R_A) \cdot i_S}{R_A + R_C} \right) = R_C \cdot \left(\frac{V_S + R_A \cdot i_S}{R_A + R_C} \right).$$

These can be seen to be the same as those obtained from Problem 1 and Problem 2 using superposition.